USE OF THE CELLULAR MODEL IN CALCULATING THE VISCOSITY OF DISPERSE SYSTEMS

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In analyzing processes in concentrated disperse systems it is not possible to disregard the effect of the perturbations introduced by each particle into the hydrodynamic field of the dispersion medium on the motion of the remaining particles. This effect can be taken into account within the framework of the "cellular" model, which is frequently used to investigate disperse systems and also in the kinetic theory of dense gases [1-3]. At high concentrations only the interaction between a particle and its neighbors is important, i.e., the hydrodynamic interaction between an individual particle and other more remote particles is effectively "screened out." According to the cellular model to each spherical particle of radius a there corresponds a hypothetical surface ("cell"), which in the first approximation may be regarded as a concentric sphere of radius $b > a$, flow perturbations created by the remaining particles affecting the flow over an individual particle only through the agency of the boundary conditions at the cell surface. In what follows this model is used as a basis for obtaining expressions for the effective coefficient of viscosity of disperse systems for various forms of the boundary conditions at the cell surface differing from those employed in [4]. As distinct from [1], where suspensions of solid particles were considered, the results obtained here relate to the case in which the inclusions are not solid.

Let a spherical particle be introduced into a flow of incompressible fluid described by the velocity distribution $v_i^{(0)} = \alpha_{ij}x_j$, where α_{ij} is the constant symmetric velocity gradient tensor, whose trace $\alpha_{ii} = 0$ by virtue of the continuity equation [5]. The corresponding solution of the hydrodynamic problem for Stokes flow over a liquid particle has the form [4]

$$\begin{gathered}\mathbf{v}^{(1)} = (2A + Br^{-5} + 5Cr^{-7})(\mathbf{v}^{(0)}\mathbf{r})\,\mathbf{r} + (-5Ar^2 - 2Cr^{-5} + D)\,\mathbf{v}^{(0)} \\ \mathbf{v}^{(2)} = 2A'(\mathbf{v}^{(0)}\mathbf{r})\,\mathbf{r} + (-5A'r^2 + D')\,\mathbf{v}^{(0)} \\ p^{(1)} = \mu_1(-21A + 2Br^{-5})(\mathbf{v}^{(0)}\mathbf{r}), \qquad p^{(2)} = -21\mu_2 A'(\mathbf{v}^{(0)}\mathbf{r}) + p_\sigma\end{gathered} \tag{1}$$

(the center of the particle is located at the coordinate origin). Here $\mathbf{v}^{(1)}$, $p^{(1)}$, μ_1 and $\mathbf{v}^{(2)}$, $p^{(2)}$, μ_2 are the velocity, pressure, and viscosity outside and inside the particle, respectively; A, B, C, D, A', D' are arbitrary constants; the significance of the quantity p_σ is discussed in detail in [4].

Keeping in mind the stress equations of motion

$$\partial\sigma_{ij} / \partial x_j = 0,$$

both outside and inside the particle, we employ the identity

$$\sigma_{ik} = \partial(\sigma_{ij}x_k) / \partial x_j,$$

which enables us to use the Gauss divergence theorem in calculating the value of the stress tensor averaged over the volume of the system $\langle\sigma_{ik}\rangle$

$$\langle\sigma_{ik}\rangle = \frac{1}{V_0}\int\limits_{V_0}\sigma_{ik}\,dV = \frac{3}{4\pi b^4}\oint\limits_{r=b}\sigma_{ij}^{(1)}x_j x_k\,dS = \alpha_{ik}\mu_1(-8.4\,Ab^2 - 1.2Bb^{-3} + 2D) \tag{2}$$

where V_0 is the volume of the cell. (Here we have made use of the continuity of the vector $\sigma_{ij}x_j$ across the surface of the particle $r = a$.) By direct calculation we obtain

Moscow. Translated from Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, Vol. 11, No. 1, pp. 183-185, January-February, 1970. Original article submitted September 18, 1969.

$$\left\langle \frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i} \right\rangle = \frac{1}{V_0} \int\limits_{V_0} \left(\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i} \right) dV = \alpha_{ik} (-8.4Ab^2 + 0.8Bb^{-3} + 2D) .$$

From the last two expressions there follows the equation for the scalar effective viscosity of the disperse system

$$\mu = \mu_1 \left[1 + 5B / (21Ab^5 - 2B - 5Db^3)\right] . \tag{3}$$

(We note that in [4] an expression somewhat different from (2) was obtained as a result of the incorrect averaging of the stress tensor.)

At the surface of the particle the following conditions should be satisfied: continuity of the velocity vector, disappearance of the normal components of the velocities, and continuity of the tangential and normal stresses. The quantity p_σ from (1) enters into only the last of these conditions (for further details see [4]).

Regarding the boundary condition at the surface of the cell r = b various opinions associated with different attitudes to the cellular model are expressed in the literature. We will consider three approaches to the problem.

1. Let $v^{(1)} = v^{(0)}$ at the surface of the cell. This condition has been used, for example, by Simha [1] in calculating the viscosity of concentrated suspensions of spherical particles. Resorting to this condition to close the algebraic system of equations for the unknown constants and introducing $\varkappa = \mu_2/\mu_1$, $\xi = a/b$, we obtain

$$\mu(\xi, \varkappa) = \mu_1 \left[1 + \xi^3 \frac{10\varkappa + 4 - 10\xi^7(\varkappa - 1)}{4(\varkappa + 1) - 5\xi^3(5\varkappa + 2) + 42\xi^5\varkappa - 5\xi^7(5\varkappa - 2) + 4\xi^{10}(\varkappa - 1)}\right]. \tag{4}$$

In the limit as $\varkappa \to \infty$, this expression takes the form:

$$\mu(\xi) = \mu_1 \left[1 + \xi^3 \frac{10(1 - \xi^7)}{4 - 25\xi^3 + 42\xi^5 - 25\xi^7 + 4\xi^{10}}\right] \tag{5}$$

which coincides with the expression obtained by Simha. In the limit as $\varkappa \to 0$, which corresponds to gas bubbles, we have

$$\mu(\xi) = \mu_1 \left[1 + \xi^3 \frac{2 + 5\xi^7}{2 - 5\xi^3 + 5\xi^7 - 2\xi^{10}}\right]. \tag{6}$$

2. Let the scalar condition $v_r^{(1)} = v_r^{(0)}$ be satisfied at the cell surface and, moreover, let there be no shear stresses between neighboring cells caused by perturbations of the main stream introduced by the presence of the particles [2].

In this case we obtain

$$\mu(\xi, \varkappa) = \mu_1 \left[1 + \xi^3 \frac{25\varkappa + 10 + 10\xi^7(\varkappa - 1)}{10(\varkappa + 1) - 2\xi^3(5\varkappa + 2) - 21\xi^5\varkappa + 5\xi^7(5\varkappa - 2) - 4\xi^{10}(\varkappa - 1)}\right] \tag{7}$$

As $\varkappa \to \infty$

$$\mu(\xi) = \mu_1 \left[1 + \xi^3 \frac{25 + 10\xi^7}{10(1 - \xi^3) - 21\xi^5 + 25\xi^7 - 4\xi^{10}}\right]. \tag{8}$$

As $\varkappa \to 0$

$$\mu(\xi) = \mu_1 \left[1 + \xi^3 \frac{5}{5 - 2\xi^3}\right] . \tag{9}$$

3. Thus, two conditions are imposed at the surface of the cell. Of these the more natural from the physical standpoint is the condition that the normal component of the velocity perturbation vanishes. In [4] an attempt was made to make do with precisely this single boundary condition at r = b. However, in this

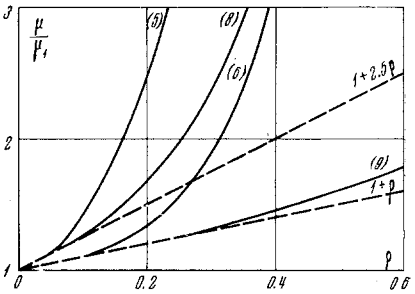

Fig. 1

case the number of constants figuring in (1) exceeds by one the number of boundary conditions. In [4] in order to overcome this difficulty it was required that the solution of the linear hydrodynamic problem of the disturbance of the main flow by a suspended particle consist of two additive parts: the undisturbed flow in the absence of a sphere $\alpha_{ij}x_j$ and the flow determined by the presence of a particle free of components proportional to the main flow. Setting, by virtue of this assumption, D = 1 everywhere in the boundary conditions and finding the constant AB, from (3) we obtain

$$\mu(\xi, \varkappa) = \mu_1 \left[1 + \xi^3 \frac{5(5\varkappa + 2) + 10\xi^7(\varkappa - 1)}{10(\varkappa + 1) + 5\xi^3(5\varkappa + 2) - 56\xi^5\varkappa + 5\xi^7(5\varkappa - 2) - 4\xi^{10}(\varkappa - 1)}\right] \tag{10}$$

As $\varkappa \to \infty$

$$\mu(\xi) = \mu_1 \left[1 + \xi^3 \frac{25 + 10\xi^7}{10 + 25\xi^3 - 56\xi^5 + 25\xi^7 - 4\xi^{10}}\right] \tag{11}$$

As $\varkappa \to 0$

$$\mu(\xi) = \mu_1 \left[1 + \xi^3 \frac{1 - \xi^7}{1 + \xi^3 - \xi^7 + 0.4\xi^{10}}\right]. \tag{12}$$

It should be noted, however, that this approach, though justified in the case of an infinite flow over a single particle, is not sufficiently well founded in the case in question.

It should be kept in mind that in Eqs. (4)-(12) the viscosity is expressed in terms of the ratio of the particle radius to the cell radius. The viscosity depends on the volume concentration of the inclusions, because the cell radius varies with concentration. Generally speaking, $b \sim a\rho^{-1/3}$; however, it is scarcely possible to specify an exact value of the cell radius that would be suitable over the entire range of concentrations, the more so in that the assumption that the cell is spherical is itself very approximate. In the first approximation it may be assumed that $b = a\rho^{-1/3}$, which corresponds to identification of the cell volume with the specific volume of a particle in the system. In this (and only this) case at small concentrations expressions (4), (7), and (10) go over into the equation obtained by Taylor for dilute emulsions of spherical droplets of one viscous liquid in another; expressions (5), (8), (11) go over into the Einstein formula, and (6), (9), (12) into the formula of Hut and Mark for a low-concentration dispersion of gas bubbles in a viscous liquid [6]. Graphs of the relations (5), (8), (6), (9), corresponding to $\xi^3 = \rho$, are presented in the Fig. 1, from which, in particular, it can be seen that the curves corresponding to case 2 coincide with the relations of Einstein and of Hut and Mark over a somewhat broader range of concentrations; however, it is difficult to decide in favor of one model rather than another.

We note that the relations obtained describe only that part of the total momentum transfer in the system that depends exclusively on the distortions of the streamlines as the liquid flows through a network of stationary particles. In reality, in disperse systems there is momentum transfer associated with the random fluctuations of the particles and the liquid. Therefore the viscosity calculated here coincides with the effective viscosity of the disperse system only when the particle fluctuations are insignificant. A more detailed discussion may be found in [4].

In conclusion it should be emphasized that the effective viscosity is determined as the proportionality factor relating the value of the stress tensor averaged over the volume of the system and the mean value of the velocity gradient tensor and not $\langle \sigma_{ik} \rangle$ and $2\alpha_{ik}$, as in [8], where the calculations of the viscosity of dilute suspensions made in [5] are repeated, but where it is unjustifiably proposed to replace the frequently confirmed term 2.5ρ in the Einstein formula by 1.5ρ.

The author thanks Yu. A. Buevich for discussing the results.

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